Log Concavity of Restricted Integer Partitions The Game

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July 2016

For the Next Eight Minutes...

What is Log Concavity Again?

Log Concavity

A sequence (a_n) is log concave iff $\forall n \; a_n^2 \geq a_{n-1}a_{n+1}$.

Which is to say...

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A sequence is log concave iff each term squared is greater than the product of the next and previous terms.

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An element a_n of a sequence (a_n) is log concave iff $a_n^2 \geq a_{n-1}a_{n+1}$.

Which is to say...

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An element of a sequence is log concave iff each term squared is greater than or equal to the product of the next and previous terms.

Log Concavity: The Work to Do Here

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- DeSalvo and Pak proved $p(n)$ is log concave for $n > 25$.
- We're going to worry about $p(n, m)$ instead.

Previously in Nichi's Presentation

For $p(n, 5)$ $n =$
5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 38, 39, 40, 41, 43, or anything higher is log concave.

For $p(n, 6)$ $n =$
6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 78, 80, 81, 82, 84, 86, or anything higher is log concave.

For $p(n, 2)$ $n =$ 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, or any other even number is log concave.

Conjecture

$\forall m \geq 2,\ r \geq 0,\ (p(m+2r, m))^2 \geq p(m+2r+1, m)p(m+2r-1, m)$

That is:
If m is even, $p(n, m)$ fulfills the LC requirement if n is even.
If m is odd, $p(n, m)$ fulfills the LC requirement if n is odd.

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  - Take the Cartesian product of the element with itself
  - Take the Cartesian product of the previous element with the next element
  - The sequence is log concave iff there's an injection from the latter to the former
  - Note if you can modify a set without changing its cardinality and get a subset, you have an injection

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An Easy Example: Adding the Rule

Remembering $p(4,2) \times p(4,2) = \{(3+1,3+1),(3+1,2+2),(2+2,3+1),(2+2,2+2)\}$ and $p(3,2) \times p(5,2) = \{(2+1,4+1),(2+1,3+2)\}$:

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- Is $\{(3+1,3+1),(3+1,2+2)\}$ a subset of $\{(3+1,3+1),(3+1,2+2),(2+2,3+1),(2+2,2+2)\}$?

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- Is $\{(3+1,3+1),(3+1,2+2)\}$ a subset of $\{(3+1,3+1),(3+1,2+2),(2+2,3+1),(2+2,2+2)\}$?
- Yes.

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We have

$$S_{2k} X S_{2k} = \left\{ \begin{array}{c} (2k-1)+1 \\ (2k-2)+2 \\ (2k-3)+3 \\ (2k-4)+4 \\ \vdots \\ (k+1)+(k-1) \\ k+k \end{array} \right\} \times \left\{ \begin{array}{c} (2k-1)+1 \\ (2k-2)+2 \\ (2k-3)+3 \\ (2k-4)+4 \\ \vdots \\ (k+1)+(k-1) \\ k+k \end{array} \right\}$$

The General Case of Two Parts

We also have

$$S_{2k-1} \times S_{2k+1} = \left\{ \begin{array}{c} (2k-2)+1 \\ (2k-3)+2 \\ (2k-4)+3 \\ (2k-5)+4 \\ \vdots \\ k+(k-1) \end{array} \right\} \times \left\{ \begin{array}{c} 2k+1 \\ (2k-1)+2 \\ (2k-2)+3 \\ (2k-3)+4 \\ \vdots \\ (k+2)+(k-1) \\ (k+1)+k \end{array} \right\}.$$

The General Case for Two Parts

Apply the Rule:

$$\left\{\begin{array}{c}(2k-1)+1\\(2k-2)+2\\(2k-3)+3\\(2k-4)+4\\\vdots\\(k+1)+(k-1)\end{array}\right\}\times\left\{\begin{array}{c}(2k-1)+1\\(2k-2)+2\\(2k-3)+3\\(2k-4)+4\\\vdots\\(k+1)+(k-1)\\k+k\end{array}\right\}.$$

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Is $\left\{\begin{array}{c}(2k-1)+1\\(2k-2)+2\\(2k-3)+3\\(2k-4)+4\\\vdots\\(k+1)+(k-1)\end{array}\right\} \times \left\{\begin{array}{c}(2k-1)+1\\(2k-2)+2\\(2k-3)+3\\(2k-4)+4\\\vdots\\(k+1)+(k-1)\\k+k\end{array}\right\}$

a subset of

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? Yes it is.

Success

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- Future Question: What rule(s) do I need to make it work?

References

- DeSalvo, Stephen, and Pak, Igor. "Log-concavity of the partition function." Ramanujan J 38 (2015): 61-73. Web. 23 Jun. 2016.
- Stanley, R. P. (1989), "Log-Concave and Unimodal Sequences in Algebra, Combinatorics, and Geometry." Annals of the New York Academy of Sciences, 576: 500–535.

Thank you

Thank you.

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