What Are Numbers?

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For the Next Twenty Minutes...

1. Defining the Integers
2. Rationals
3. The Reals are Complete
4. Complex Numbers, Quaternions, Octonions, Sedenions, etc.
5. More Number Systems to Consider
   - Hyperreals
   - Surreals
   - Wheel Theory
   - Constructive Reals
6. Conclusion

What Are Numbers?

- Numbers are used frequently in mathematics.

What Are Numbers?

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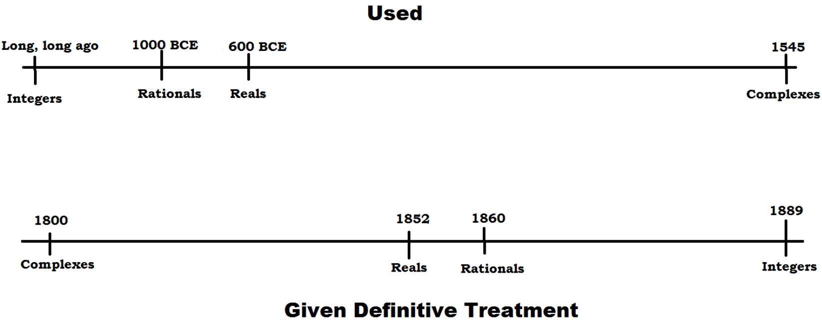
- What are they?

Caveat: Representations are Not Number Systems

The following all represent the same integer:

- ten (English)
- 10 (decimal)
- |||||||||| (tally marks)
- 1010 (binary)
- ∪ (hieroglyphics)

History


Used
Long, long ago
Integers
1000 BCE
Rationals
600 BCE
Reals
1545
Complexes
1800
Complexes
1852
Reals
1860
Rationals
1889
Integers
Given Definitive Treatment

Integers

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8. For every natural number x, $S(x) = 0$ is false.
9. If K is a set such that $0 \in K$, and for every natural number x, $x \in K$ implies that S(n) is in K, then K contains every natural number.

Fields

Definition

A set S with two operations called addition and multiplication which S is closed under is a **field** if

- addition and multiplication are commutative and associative
- multiplication distributes over addition
- there exist an additive identity, multiplicative identity, additive inverses, and multiplicative inverses

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The additive identity is 0 since $n + 0 = n = 0 + n$ for all natural n.
No natural number can be added to 2 to get 0. Thus 2 has no inverse.

Integers

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For each natural n create a negative number $-n$ such that $-n + n = 0 = n + (-n)$.

Properties of Integers

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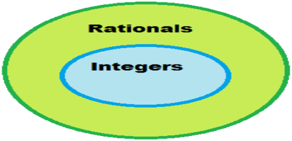
We need $1/2$, which is a rational.

Rationals Let Us Divide Anything

What if we want to divide any integers?



Examples: $\frac{1}{2}, \frac{3}{8}, -\frac{4}{9}$

Constructing the Rationals

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- **Not complete.**

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- Thus $\frac{1}{1}$ is the identity element.

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- Since $1pq = 1pq$, $\frac{1}{1} \sim \frac{p}{q}$.
- Therefore rationals have multiplicative inverses.

Completeness

Definition

A field is **Archimedean** if for any two positive a, b there is a positive integer n such that $b < na$.

Rationals are Archimedean. E.g. if you have $\frac{1}{2}$ and $\frac{1}{3}$ then you can find $\frac{1}{2} < \frac{3}{1} \cdot \frac{1}{3}$ and $\frac{1}{3} < \frac{7}{1} \cdot \frac{1}{3}$.

We will explore a system that is not Archimedean later

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A field S is **Cauchy complete** if every Cauchy sequence in S has a limit in S.

The rationals fail this. E.g. $\frac{3}{1}, \frac{31}{10}, \frac{314}{100}, \cdots$ is not approaching a rational.

Completeness

Definition

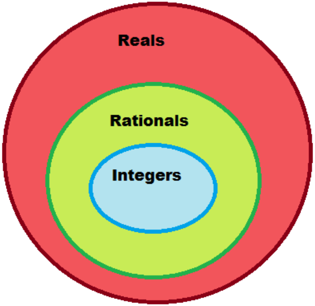
A field is **(order) complete** if it is Cauchy complete and Archimedian.

Rationals are Archimedian and not Cauchy complete.

Reals are complete.

Introducing the Reals

What if we want a complete ordered field?



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Equivalence Relation on the Reals

- Equivalence relation on the set of Cauchy sequences: Two sequences $x = \{x(n)\}$ and $y = \{y(n)\}$ are equivalent if the difference $\{x(n) - y(n)\}$ is a null sequence.

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- Therefore the two sequences are the same number.
- We can define the **real numbers** as the set of equivalence classes of Cauchy sequences of rationals.

Only Complete Ordered Field

Theorem

The reals are the only complete ordered field.

Proof The Reals are Archimedian

Definition

A field is **Archimedian** if for any two positive x, y there is a positive n such that $b < na$.

Let x and y be positive reals. Let $0 < a < x$ and a be rational since between any two reals you can pick a rational. Let $y < b$. Rationals are Archimedian. Thus there is an n such that $b < na$.
From $y < b$ and $b < an$ we have $y < an$.
From $0 < a < x$ and $y < an$ we have $y < nx$.
Therefore $y < nx$.

Proof The Reals are Cauchy Complete

Definition

A field S is **Cauchy complete** if every Cauchy sequence in S has a limit in S.

Let $\{x(n)\}$ be a Cauchy sequence with each $x(n) \in \mathbb{R}$.
Each element $x(n)$ is itself a Cauchy sequence of rationals $\{b(n)\}$.
The sequence $\{x(n)\}$ converges to some specific $\{b_0(n)\}$ which must be a Cauchy sequence of rationals.
Therefore $\{x(n)\}$ converges to a real. So the reals are Cauchy complete.

Proof The Reals are Complete

Definition

A field is complete if it is Cauchy complete and Archimedian.

The reals are Archimedian and Cauchy complete. Therefore the reals are complete.

Another Definition of Reals

Definition

A **cut** is a subset X of the rationals such that

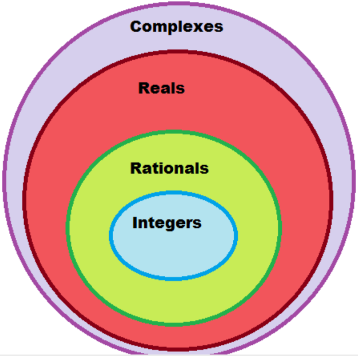
1. Neither X nor its complement is empty.
2. If r is in X and $s > r$ then s is in X.
3. X has no smallest element.

E.g. The set of positive rationals is a cut since

- it's not empty
- it's complement (zero and the negatives) is not empty
- every number bigger than a positive rational is a positive rational
- there is no smallest rational

Complex Numbers

The complex numbers are the most common extension of the real numbers.



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- And also all polynomials.
- They are two-dimensional.
- While commonly seen in the form $a + bi$, we will express them as matrices, taking advantage of matrices' existing algebraic structure.

Complex Part II

A complex number is a matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

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- Field.

Complex Part II

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- Complete*.

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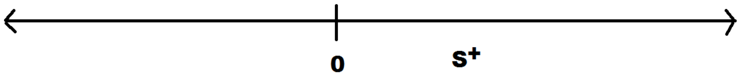
- Field.
- Complete*.
- **Not Ordered**.

Linear Order

Definition

A field S is **linearly ordered** if there is a subset $S^+ \subset S$ such that if $a, b \in S^+$ then $a + b, ab \in S^+$ and if $a \in S$ then exactly one of the following is true: $a = 0$ or $a \in S^+$ or $-a \in S^+$.



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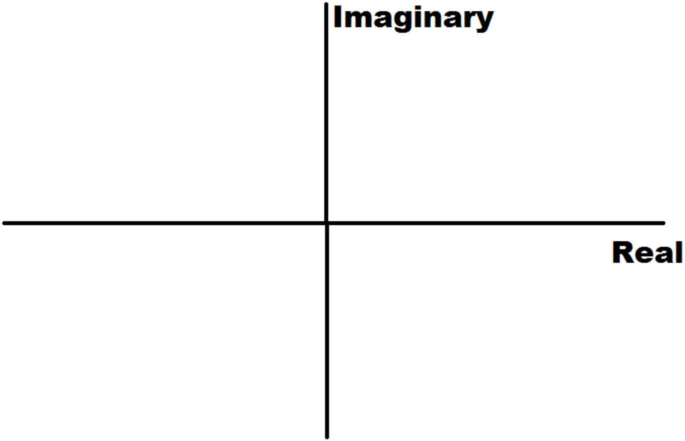

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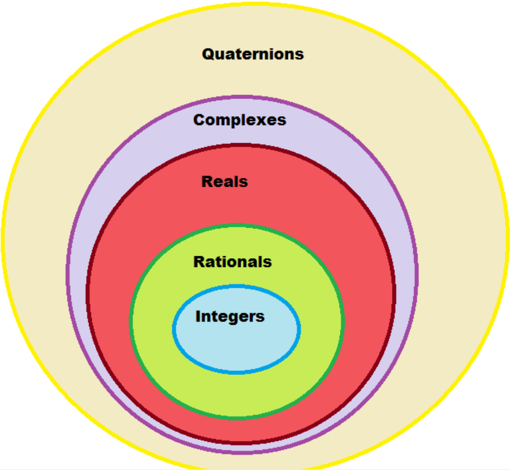
- We can have four dimensions if we give up commutativity of multiplication.
- For this we have the set of **quaternions**.
- A quaternion is represented by 4 by 4 matrices of the form

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \text{ where } a, b, c, d \in \mathbb{R}.$$

Quaternions



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- $ij = -ji = k$
- $jk = -kj = i$
- $ki = -ik = j$

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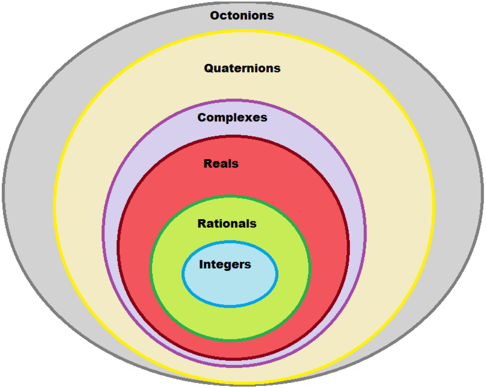
- $ki = -ik = j$
- Field minus commutativity of multiplication
  - $ij = k$
  - $ji = -k$
- Complete
- Still not ordered

Octonions

What if we want to expand our matrix again?



Octonions

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Definition

An algebra is **alternative** if for all x, y in the algebra $x(xy) = (xx)y$ and $y(xx) = (yx)x$.

Octonions

Written $x = x_0 e_o + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7$.

Octonions

Written $x = x_0 e_o + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7$.

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Sedenions

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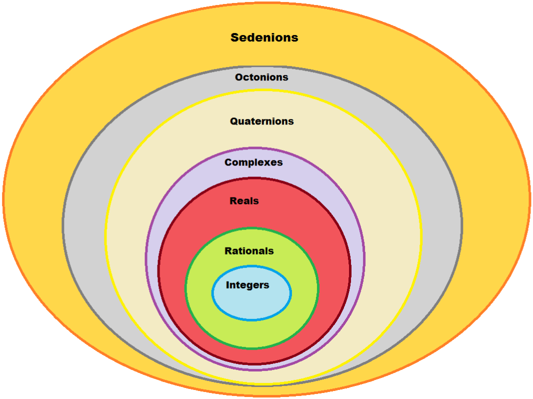
They have a zero divisor.
$(e_3 + e_{10})(e_6 - e_{15}) = 0$
Thus $0/(e_3 + e_{10}) = (e_6 - e_{15})$.

Sedenions


Sedenions
Octonions
Quaternions
Complexes
Reals
Rationals
Integers

2^n-nions

We can continue with more dimensions and less properties of numbers.


BUT WAIT
THERE'S MORE

Hyperreals

What if we want infinitely big or infinitely small numbers?

Hyperreals

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Definition

A hyperreal x is **infinite** if $|x| > r$ for all $r \in \mathbb{R}$.

Definition

A hyperreal x is **infinitesimal** if $|x| < r$ for all $r \in \mathbb{R}^+$.

Clouds

Around every real number there is a cloud the size of the real number line.

Infinites

Let w be an infinite number. Then $1+1+1+\cdots+1<w$ for any finite number of 1s.

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$w+1+1+1+\cdots+1<2w$ for any finite number of 1s.

Hyperreals Are Not Archimedian

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A field is **Archimedian** if for any two positive a, b there is a positive integer n such that $b < na$.

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Let a be an infinitesimal and b a positive real. Then for any integer n, na is an infinitesimal. Since all infinitesimals are smaller than any positive reals, $b > na$ for any integer n.

Construction of the Hyperreals

Hyperreals are constructed with sequences of reals.

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Definition

Two hyperreals $x = \{x_n\}, y = \{y_n\}$ are **equal** if $\{n | x_n = y_n\}$ is a big set.

e.g. $\{1, 2, 2, 2, 2, \ldots\} = \{3, 4, 2, 2, 2, 2, 2, \ldots\}$

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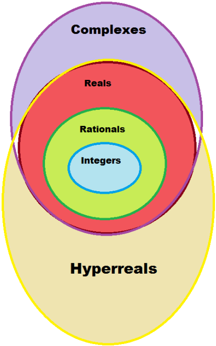
Definition

A hyperreal $x = \{x_n\}$ is positive if $\{n | x_n > 0\}$ is a big set.

e.g. $\{0, -1, -3, 1, 1, 1, 1, 1, 1, 1, 1, \ldots\}$ is positive

Properties of the Hyperreals

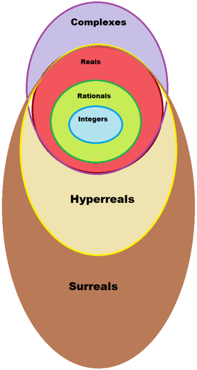


All properties of real numbers (not necessarily sets thereof) transfer to the hyperreals.

Surreals

What if we want a number system that is even more general than the hyperreals, generates numbers all individually by induction, introduces proof by playing a game, and doesn't even require assuming the integers exist to construct?

Surreals



Surreals

Definition

A **surreal number** is a game x in which all the options of x are numbers, and no inequality of the type $x^L \geq x^R$ occurs.

The Surreal Number Game

Rule: The left and right must share no elements
Rule: Every element on the right must be greater than every element on the left.

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On day zero the surreal number $0 = \{|\}$ is born.

On the next few days these are born:
$1 = \{0|\}$ $2 = \{0, 1|\}$ $3 = \{0, 1, 2|\}$

$-1 = \{|0\}$ $-2 = \{|0, -1\}$ $-3 = \{|0, -1, -2\}$

$\frac{1}{2} = \{0|1\}$ $\frac{1}{4} = \{0|\frac{1}{2}\}$ $\frac{3}{4} = \{\frac{1}{2}|1\}$

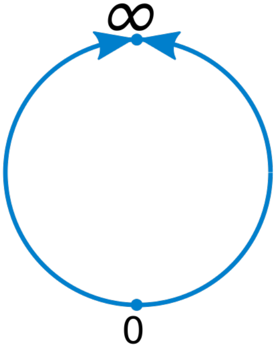
Projectively Extended Real Line

What if we want to divide by zero?

Projectively Extended Real Line

What if we want to divide by zero?

Tape the ends of the real line together.



PERL is a Wheel

The PERL is not a field, but it is a **wheel**.

Definition

Wheels are a type of algebra where division is always defined.

Under a wheel division is a unary operator instead of a binary operator.
I.e. a/b is shorthand for $a \cdot /b$.

Fun Properties of Wheels

- $x/0 = \infty$
- In general $0x \neq 0$
- In general $x - x \neq 0$
- In general $x/x \neq 1$
- $x - x = 0x^2$
- $x/x = 1 + 0x/x$
- $0/0 + x = 0/0$

Constructive Reals

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Constructive Reals

- Mathematical constructivists don't like pure existence proofs.
- You have to construct something to show it exists.
- No law of excluded middle nor double negation is allowed as a rule of inference.
- We can still get a real number system except they are not provably (order) complete.
- A **constructive real number** is a sequence $x = \{x_n\}$ of rational numbers such that $|x_n - x_m| < \frac{1}{m} + \frac{1}{n}$.

A Few More Options

A Few More Options

- bicomplexes
- multicomplexes
- split-quaternions
- biquaternions
- surcomplexes
- supernaturals
- split-octonions
- dual numbers
- split-biquaternions
- superreals
- fuzzy numbers
- p-adics

Conclusion

We have some options for what numbers are.

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- Sequences
- Equivalence classes
- Games
- Matrices
- Other things

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- Mental constructions
- Just symbol manipulations

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What things qualify as numbers comes up to the situation and utility. But this illuminates the first question.

References

- Henle Henle, Michael, Which Numbers are Real? The Mathematical Association of America, 2012.
- Russell, Bertrand. Introduction to Mathematical Philosophy. 5 February 2010. Web. 6 May 2016. http://people.umass.edu/klement/imp/imp-ebk.pdf
- Kline, Morris. Mathematical Thought from Ancient to Modern Times, Oxford University Press, 1990.
- Burril, Claude W., Foundations of Real Numbers, McGraw-Hill Inc., 1967.

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